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Motion of Decaying Vortex Rings with Non-Similar Vorticity Distributions*

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SUMMARY

The motion and decay of circular vortex rings with an inner viscous core is considered by systematic matching of inner and outer asymptotic expansions. The governing Navier–Stokes equations are reduced to a coupled integro-differential system. A method of construction of solutions for the integro-differential system is presented. The initial vorticity distribution may be non-similar. Also presented is a method for introducing a time shift which makes the first term in the series solution for the vorticity to be the "best" approximation. The analysis is then applied to the motion and decay of a pair of coaxial vortex rings.

1. Introduction

The motion and decay of circular vortex rings with non-similar vorticity distributions submerged in an inviscid stream is being considered. The geometry of the ring is presented in Fig. 1. The radius of the ring is denoted by R, the position of the center of the ring is Z, the effective radius of the vortical core is δ, and the ring has a circulation Γ. A toroidal coordinate system (r, θ) is attached to the center of the vortical core. The flow field in the absence of the ring is assumed to be a given axially symmetric inviscid flow.

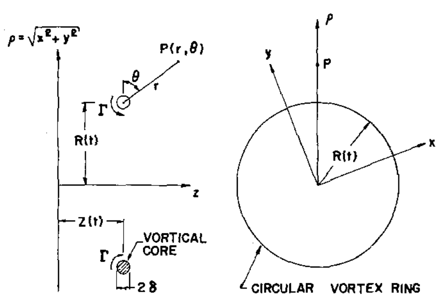

Figure 1. Geometry of vortex ring.

The classical inviscid theory for the motion of circular vortex rings (e.g. see [1]) contains some serious shortcomings [2]. These include: (i) the velocity at the center of the vortical core ($r=0$) is infinite; (ii) if the axial velocity of the ring becomes infinite as the size of the vortical core tends to zero ($\delta \to 0$); (iii) $\delta \neq 0$, it must then be arbitrarily assigned in order to define the velocity of the ring; and (iv) the viscous effect is ignored even though the velocity gradient in the core is large.

C. Tung and L. Ting [2] recognized that the assumption of vanishing viscous force becomes invalid near the center of the vortical core where velocities and radial derivatives are large. Therefore they divided the flow field into two regions, an inner region near the center of the vortical core where viscous forces are important, and an outer region away from the vortical core where the flow field is inviscid. Tung and Ting then introduced asymptotic expansions for

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the vorticity and the stream function Ψ in both the inner viscous region and the outer inviscid region. The expansion parameter $\varepsilon = 1/\mathrm{Re}^{\frac{1}{2}}$ was used, where the Reynolds number Re is defined as the ratio of the circulation Γ to the kinematic viscosity ν. By the technique of systematic matching [3] Tung and Ting derived a coupled system of partial differential equations in space and time and ordinary differential equations in time to govern the decay of the vorticity, the flow field and the motion of the vortex ring. In order to uncouple the partial differential equation for the decay behavior from the others, Tung and Ting assumed that the vorticity distribution is similar. This implied a restriction on the initial data. They then used this vorticity distribution to find equations for the axial and radial motion of the ring.

In the present paper no restriction is imposed on the initial vorticity distribution. The solution for the partial differential equation for the flow field and the matching condition are combined to yield an integro-differential equation in t for the axial position of the ring. The partial differential equation for the vorticity is uncoupled by the introduction of new variables in section 3. This equation is then independent of both the outer inviscid solution and the instantaneous position of the ring. A solution for the vorticity as a series of eigenfunctions is then obtained. The coefficients in the series are related to the initial data. These solutions are then combined with the remaining ordinary differential equations to form a system of integro-differential equations in t for the motion of the ring. This system is then integrated numerically for a given initial position of the ring.

Inspection of the partial differential equations for the vorticity yields that the equation is independent of the initial value of the new variables (see above), and therefore independent of a translation in these variables. A shift is chosen in section 5 which makes the second term in the series for the vorticity distribution vanish. This is equivalent to the "optimum solution" introduced by Ting and Chen [4] for boundary layer theory and subsequently used by Kleinstein and Ting [5] for heat conduction problems and by Ting [6] for two-dimensional vortices. The accuracy of one term optimum solutions as compared to the series solution is demonstrated in the study of the motion and decay of two co-axial vortex rings.

2. Equations for the Motion and Decay of the Ring

The governing equations for the leading terms for the motion and decay of circular vortex rings were first derived by Tung and Ting [2] and subsequently modified for the general three-dimensional case by Ting [6]. In this section are presented an outline of the derivation of those equations with slight modifications in order to facilitate the subsequent analyses.

The basic governing equations for the motion and decay of circular vortex rings are given by the Navier–Stokes equations. In the outer region, the flow is irrotational ($\zeta = 0$) and the stream function Ψ is expanded in a power series in ε:

$$\Psi(t, \rho, z, \varepsilon) = \Psi^{(0)}(t, \rho, z) + \varepsilon \Psi^{(1)}(t, \rho, z) + \dots \,.$$

In the inner region, the radial variable is stretched, i.e. $\bar{r} = r/\varepsilon$. The stream function Ψ and the vorticity $\bar{\zeta}$ in the inner region are also expanded in power series in ε:

$$\bar{\Psi}(t, \bar{r}, \theta, \varepsilon) = \bar{\Psi}^{(0)}(t, \bar{r}) + \varepsilon \bar{\Psi}^{(1)}(t, \bar{r}, \theta) + \dots$$

and

$$\bar{\zeta}(t, \bar{r}, \theta, \varepsilon) = \varepsilon^{-2} \bar{\zeta}^{(0)}(t, \bar{r}) + \varepsilon^{-1} \bar{\zeta}^{(1)}(t, \bar{r}, \theta) + \dots \,.$$

Substituting the above expansions into the Navier–Stokes equations, the following equations governing the two leading terms in $\bar{\Psi}$ and $\bar{\zeta}$ are obtained:

$$\bar{r}\bar{\zeta}^{(0)} = -(\bar{\Psi}_r^{(0)}/\bar{r})_{\bar{r}}\,; \quad \bar{r}\bar{\zeta}^{(1)} = -(\bar{\Psi}_r^{(1)}/\bar{r})_{\bar{r}}\,, \tag{2.1}$$

$$\bar{r}\bar{\zeta}_t^{(0)} - \Gamma[\bar{r}\bar{\zeta}_r^{(0)}]_{\bar{r}} = (\dot{R}_0/2R_0)[\bar{r}^2\bar{\zeta}^{(0)}]_{\bar{r}}\,, \quad \text{and} \tag{2.2}$$

$$\bar{\Psi}_\theta^{(1)}\bar{\zeta}_{\bar{r}}^{(0)} - \bar{\Psi}_{\bar{r}}^{(0)}\bar{\zeta}_\theta^{(1)} = [\bar{r}\bar{\zeta}^{(0)}/R_0]\,\bar{\Psi}_r^{(0)} \sin\theta\,. \tag{2.3}$$

In Eqs. (2.1) through (2.3), $\bar{\Psi}$ is the stream function of the velocity components $\bar{u}$, $\bar{v}$ relative to the center of the vortical core. In Eqs. (2.1) and (2.3), $\tilde{\Psi}^{(1)}$ and $\tilde{\zeta}^{(1)}$ are the parts of $\bar{\Psi}^{(1)}$ and $\bar{\zeta}^{(1)}$ which depend on θ. In Eq. (2.3) R_0 is the leading term in the expansion of $R(t, \varepsilon)$ in a power series in ε. The boundary conditions are given by the regularity conditions at $\bar{r}=0$:

$$\bar{\Psi}_{\bar{r}}^{(0)}(t, \bar{r}=0)=0\,,\quad \bar{\zeta}^{(0)}(t, \bar{r}=0)=\text{finite}\,, \tag{2.4}$$

$$\tilde{\Psi}_{\bar{r}}^{(1)}(t, \bar{r}=0)=0 \text{ and } \lim_{\bar{r}\to 0}\left[\tilde{\Psi}_{\theta}^{(1)}(t, \bar{r}, \theta)/\bar{r}\right]=0\,, \tag{2.5}$$

and by matching the outer solution as $r\to 0$ with the inner solution as $\bar{r}\to\infty$:

$$\bar{v}^{(0)}=-\bar{\Psi}_{\bar{r}}^{(0)}/R_0\to\Gamma/2\pi\bar{r} \tag{2.6}$$

$$\frac{1}{\bar{r}}\tilde{\Psi}^{(1)}(t, \bar{r}, \theta)\to\frac{\Gamma}{4\pi}\cos\theta\left[\ln\left(\frac{\delta R_0}{r}\right)-1\right]+R_0\left[-(w_1^*-\dot{R}_0)\sin\theta+(w_3^*-\dot{Z}_0)\cos\theta\right]. \tag{2.7}$$

In Eq. (2.7) Z_0 is the leading term in the expansion of $Z(t, \varepsilon)$ in a power series in ε, $\dot{Z}_0=dZ_0/dt$, $\dot{R}_0=dR_0/dt$, and w_1^* and w_3^* are velocity components along the (ρ, z) axes evaluated at (R_0, Z_0) without the contribution of the vortex ring.

If $\tilde{\Psi}^{(1)}(t, \bar{r}, \theta)$ is resolved into Fourier components in θ, Eq. (2.3) reduces to:

$$\frac{\partial^2\tilde{\Psi}_{jk}^{(1)}}{\partial\bar{r}^2}+\frac{1}{\bar{r}}\frac{\partial\tilde{\Psi}_{jk}^{(1)}}{\partial\bar{r}}-\left[\frac{1}{\bar{r}^2}-\frac{R_0\bar{\zeta}_{\bar{r}}^{(0)}}{\bar{\Psi}_{\bar{r}}^{(0)}}\right]\tilde{\Psi}_{jk}^{(1)}=\sigma_{jk}\left[2\bar{r}\bar{\zeta}^{(0)}-\bar{v}^{(0)}\right] \tag{2.8}$$

where $j=1, 2, 3\ldots$, $k=1$ or 2, $\sigma_{11}=1$ and $\sigma_{jk}=0$ otherwise. The regularity conditions at $\bar{r}=0$ and the matching conditions at $\bar{r}\to\infty$ yields for $\tilde{\Psi}_{jk}^{(1)}$, $j\neq 1$, governed by Eq. (2.8):

$$\tilde{\Psi}_{jk}^{(1)}(t, \bar{r})=0 \quad\text{for}\quad k=1 \text{ or } 2\,,\qquad j=2, 3, 4\ldots\,.$$

The regularity conditions at $\bar{r}=0$ are enough to show that $\tilde{\Psi}_{12}^{(1)}(t, \bar{r})=0$ and

$$\tilde{\Psi}_{11}^{(1)}(t, \bar{r})=\frac{\bar{\Psi}_{\bar{r}}^{(0)}}{R_0}\int_0^{\bar{r}}\{\xi[\bar{\Psi}_{\xi}^{(0)}]^2\}^{-1}\int_0^{\xi}\frac{1}{s}\frac{\partial}{\partial s}\{s^3[\bar{\Psi}_s^{(0)}]^2\}\,ds\,d\xi\,. \tag{2.9}$$

Then the matching conditions as $r\to\infty$ yield:

$$\dot{R}_0(t)=w_1^*(t, R_0, Z_0) \tag{2.10}$$

and

$$\dot{Z}_0(t)=w_3^*(t, R_0, Z_0)+\frac{\Gamma}{2\pi R_0}\left\{\ln\left[\frac{8R_0}{r}\right]-1\right\}-\lim_{\bar{r}\to\infty}\frac{1}{R_0\bar{r}}\tilde{\Psi}_{11}^{(1)}(t, \bar{r})\,. \tag{2.11}$$

For large $\bar{r}$, Eq. (2.11) reduces to:

$$\frac{1}{\bar{r}}\tilde{\Psi}_{11}(t, \bar{r})=-\frac{\Gamma}{4\pi}\left\{\ln\bar{r}+\tfrac{1}{2}-\frac{4\pi^2}{\Gamma^2}\int_0^{\infty}\frac{\partial}{\partial\xi}[\xi^2\bar{v}^{(0)}]^2\ln\xi\,d\xi\right\}+O\left(\frac{1}{\bar{r}^2}\right). \tag{2.12}$$

Substituting Eq. (2.12) into Eq. (2.11) results in:

$$\dot{Z}_0(t)=w_3^*(t, R_0, Z_0)+\frac{\Gamma}{4\pi R_0}\left\{\ln\left[\frac{8R_0}{\varepsilon}\right]-\tfrac{1}{2}-K(t)\right\} \tag{2.13}$$

where

$$K(t)=4\pi^2/\Gamma^2\int_0^{\infty}\frac{\partial}{\partial\xi}\{[\xi\bar{v}^{(0)}(t,\xi)]^2\}\ln\xi\,d\xi\,. \tag{2.14}$$

Eqs. (2.2), (2.10) and (2.13) are the equations governing the leading terms in the motion and decay of a circular vortex ring. The equations are coupled and make up an integro-differential system for $\bar{\zeta}^{(0)}(t, \bar{r})$, $R_0(t)$ and $Z_0(t)$. Initial conditions on the three dependent variables are to be freely prescribed.

The vorticity in the outer region is identically zero. Therefore matching $\bar{\zeta}^{(0)}(t, \bar{r})$ as $\bar{r}\to\infty$ with the zero vorticity in the outer region requires that

$$\bar{\zeta}^{(0)}(\bar{r}, t)=o\,(1/\bar{r}^{\alpha}) \tag{2.15}$$

for large $\bar{r}$, where α is any positive number. Eq. (2.15) holds because if $\bar{\zeta}^{(0)}(\bar{r}, t)=O(1/\bar{r}^k)$, where k is finite, then $\zeta^{(0)}(\bar{r}, t)=O(\varepsilon^k/r^k)$, and would have to match with a non-zero outer region vorticity. Integrating Eq. (2.2) over the whole space yields that the total space integral of $\bar{\zeta}^{(0)}$ is invariant, i.e.:

$$\int_0^\infty \bar{\zeta}^{(0)}(\bar{r}, t)(2\pi\bar{r})\,d\bar{r} = \Gamma\,. \tag{2.16}$$

Then, since $(\bar{v}^{(0)}\bar{r})_{\bar{r}}=\bar{r}\bar{\zeta}^{(0)}$,

$$\bar{v}^{(0)} = \frac{1}{\bar{r}}\int_0^{\bar{r}} \zeta^{(0)}(\bar{r}, t)\,d\bar{r} \tag{2.17}$$

and Eq. (2.16) implies that $\bar{v}^{(0)}\to\Gamma/2\pi\bar{r}$ as $\bar{r}\to\infty$, which is the matching condition Eq. (2.6). Thus the matching condition is satisfied independently of the functional form of the vorticity distribution.

Eq. (2.15) implies that $\bar{\zeta}^{(0)}$ decays exponentially with respect to $\bar{r}$. It will be seen in the next section that this condition implies that eigenvalues obtained in solving the transformed vorticity equations are discrete.

3. Solution for the Decay of Ring

Tung and Ting [2] were able to construct similarly solutions of Eq. (2.2) by the change of variable t to τ_2 where:

$$\tau_2 = \int_0^t dt'\,R_0(t')/R_0(t)\,. \tag{3.1}$$

The similarity variable is then defined to be $\eta=\bar{r}/(4\Gamma\tau_2)^{\frac{1}{2}}$ and the vorticity distribution is identified as the two-dimensional similarity solution. For two-dimensional vortices (for which $\tau_2=t$) Ting [6] constructed non-similar solutions for $\zeta^{(0)}(t, \eta)$ by separation of variables. The result was a series solution of $\bar{\zeta}^{(0)}$ in terms of eigenfunctions. This procedure cannot be directly followed for vortex rings because the term on the right-hand side of Eq. (2.2) does not allow for a variable separable solution for $\bar{\zeta}^{(0)}(\tau_2, \eta)$.

By introducing a new time variable τ_1 defined by

$$\frac{d\tau_1}{dt} = R_0(t)\,,\quad \tau_1(t_0)>0 \tag{3.2}$$

and redefining τ_2 as

$$\tau_2 = \tau_1/R_0(t) \tag{3.3}$$

a variable separable solution of Eq. (2.2) can be found. As a first step, let $\bar{\zeta}^{(0)}(t, \bar{r}) = f(\tau_1, \eta)/\tau_2$ where $\eta=\bar{r}/(4\Gamma\tau_2)^{\frac{1}{2}}$ in Eq. (2.2):

$$\tau_1 f_{\tau_1} = \{(\eta f_\eta)_\eta + (2\eta^2 f)_\eta\}/4\eta\,. \tag{3.4}$$

Eq. (3.4) is independent of R_0 and is therefore uncoupled from the ordinary differential equations for the position of the ring. Moreover, Eq. (3.4) is amenable to the method of separation of variables. Letting $f(\tau_1, \eta)=F(\tau_1)G(\eta)$ in Eq. (3.4) results in:

$$F'\tau_1 + \lambda F = 0 \tag{3.5}$$

and

$$(\eta G')' + 2\eta^2 G' + 4(\lambda+1)\eta G = 0\,. \tag{3.6}$$

Letting $\xi=\eta^2$ and $B(\xi)=e^{-\xi}G(\sqrt{\xi})$ in Eq. (3.6) yields:

$$\xi B'' + (1-\xi)B' + \lambda B = 0\,. \tag{3.7}$$

The condition of exponential decay on $\bar{\zeta}^{(0)}$, Eq. (2.15), implies that G must decay exponentially with respect to η, and therefore B cannot increase exponentially with respect to ξ. Eq. (3.7) admits two type of solutions [7], one having continuous eigenvalues and displaying exponential growth, the second having discrete eigenvalues $\lambda = n = 0, 1, 2 \ldots$. With $\lambda = n$, Eq. (3.7) reduces to Laguerre's Equation and therefore $B(\xi) = L_n(\xi)$ where $L_n(\xi)$ is the nth Laguerre polynomial:

$$L_n(\xi) = n! \sum_{m=0}^{n} (-1)^m \xi^m / [(n-m)!\, m!\, m!] \,. \tag{3.8}$$

With the restriction of Eq. (2.15), the second type solution for $B(\xi)$ is the only admissible one. Then the solution of Eq. (3.6) is given by $G_n = \exp(-\eta^2) L_n(\eta^2)$. The solution of Eq. (3.5) is then $F = 1/\tau_1^n$, and therefore

$$\bar{\zeta}^{(0)}(t, \bar{r}) = \frac{e^{-\eta^2}}{\tau_2} \sum_{n=0}^{\infty} \frac{C_n L_n(\eta^2)}{\tau_1^n} \,. \tag{3.9}$$

The coefficients C_n are related to the specified initial data $\bar{\zeta}^{(0)}(t_0, \bar{r})$ by [5]

$$C_n = \tau_{20} \tau_{10}^n \int_0^\infty L_n(\eta^2) \bar{\zeta}^{(0)} [(t_0, \eta (4\Gamma \tau_{20})^{\frac{1}{2}}]\, d(\eta^2) \tag{3.10}$$

where τ_{10} and τ_{20} and τ_1 and τ_2 evaluated at $t = t_0$, respectively.

Eqs. (3.8), (3.9), and (3.10) complete the solution for the vorticity $\bar{\zeta}^{(0)}(t, \bar{r})$ for a given initial vorticity distribution $\bar{\zeta}^{(0)}(t_0, \bar{r})$. Eqs. (2.10) and (2.13) are then the resulting system for the position of the ring, R_0 and Z_0, which is discussed in the next section. The initial values τ_{10} and τ_{20} are discussed in section [5].

4. The Governing System for the Motion of the Ring

In the previous section the decay behavior of circular vortex rings was obtained independently of the solution for the position of the ring. The governing equations for $R_0(t)$ and $Z_0(t)$ are Eqs. (2.10) and (2.13). The function $K(t)$ appearing in Eq. (2.13) and given by Eq. (2.14) depends on $\bar{v}^{(0)}(t, \bar{r})$ which in turn can be related to $\bar{\zeta}^{(0)}(t, \bar{r})$ by Eq. (2.17). Therefore, since $\bar{\zeta}^{(0)}$ is now determined, $K(t)$ is known.

If Eq. (3.1) is differentiated with respect to time, a differential equation for τ_2 is obtained, i.e.

$$\dot{\tau}_2 = 1 - \tau_2 / R_0 \,. \tag{4.1}$$

Eqs. (2.10), (2.13) and (4.11) form a first order differential system for the dependent variables R_0, Z_0, and τ_2 with respect to the independent variable t. Initial conditions on R_0 and Z_0 are prescribed at $t = t_0$. The initial conditions for τ_2 will be discussed in the next section.

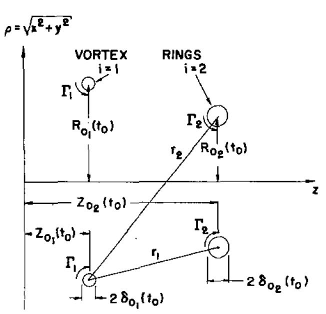

Figure 2. Initial geometry for two coaxial vortex rings.

The differential system is now ready to be integrated numerically for a given initial position of a vortex ring submerged in a given axially symmetric inviscid flow, i.e. given w_1^* and w_3^*. Fig. 3 shows the results of the above numerical integration for the special example described in section (6).

5. Optimum Asymptotic Solution

Integrating Eq. (3.2) yields

$$\tau_1 = \int_{t_0}^{t} R(t')\,dt' + \tau_{10} \tag{5.1}$$

where $\tau_{10} = \tau_1(t_0)$ is positive but otherwise arbitrary. The restriction $\tau_{10} > 0$ is necessary to obtain series solutions for the vorticity of the type discussed in section (3). The arbitrariness of τ_{10} is due to that in deriving Eq. (3.4) only Eqs. (3.2) and (3.3) were needed, and not Eq. (5.1). Combining Eqs. (3.3) and (5.1) yields

$$\tau_2 = \left\{ \int_{t_0}^{t} R(t')\,dt' + \tau_{10} \right\} \Big/ R(t) \tag{5.2}$$

and therefore:

$$\tau_{20} = \tau_2(t_0) = \tau_{10}/R(t_0)\,. \tag{5.3}$$

Since τ_{10} is arbitrary, a convenient way of choosing it is to make the first term in the series for $\bar{\zeta}^{(0)}$, Eq. (3.9), approximate $\bar{\zeta}^{(0)}$ as well as possible, i.e. setting $C_1 = 0$. This results in

$$\tau_{20}^{*} = \frac{1}{4\Gamma^2} \int_0^{\infty} [\bar{r}^2 \bar{\zeta}^{(0)}(t_0, \bar{r})](2\pi \bar{r}\, d\bar{r}) \tag{5.4}$$

and therefore the optimum shift for the variable τ_{10} is

$$\tau_{10}^{*} = [R_0(t_0)/4\Gamma^2] \int_0^{\infty} [\bar{r}^2 \bar{\zeta}^{(0)}(t_0, \bar{r})](2\pi \bar{r}\, d\bar{r}) \tag{5.5}$$

where $\bar{\zeta}^{(0)}(t_0, \bar{r})$ is the given initial vorticity distribution.

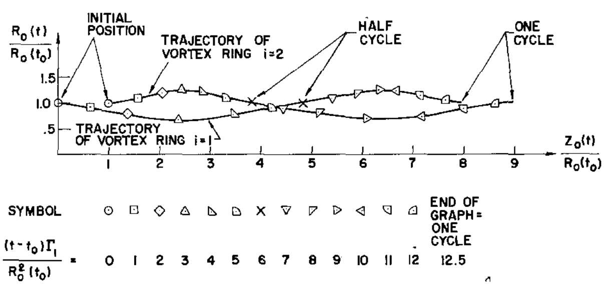

Figure 3. Trajectory of a pair of coaxial vortex rings.

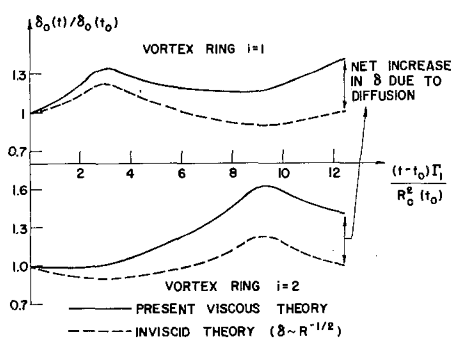

Figure 4. Vortical core sizes vs. time for one cycle.

The integral of the one term optimum solution matches the integral of the initial profile regardless of the choice of τ_{20} because setting the above two integrals equal to each other recovers the definition of C_0, Eq. (3.10) with $n = 0$. The one term solution for $\bar{\zeta}^{(0)}$ is given from Eqs. (3.9) by $C_0\, e^{-\eta^2}/\tau_2$, and therefore

$$\int_0^\infty \zeta^{(0)}(t_0, \bar{r})\bar{r}\,d\bar{r} = \frac{C_0}{\tau_2}\int_0^\infty e^{-\eta^2}\bar{r}\,d\bar{r}\,.$$

Then

$$C_0 = \tau_{20}\int_0^\infty \bar{\zeta}^{(0)}(t_0, \bar{r})\frac{\bar{r}\,d\bar{r}}{2\Gamma\tau_{20}} \tag{5.6}$$

which is indeed the same as Eq. (3.10) with $n=0$. In general the second moment of a one term solution for $\bar{\zeta}^{(0)}$ evaluated at t_0 will not match the second moment of the initial profile. However, if the optimum one term solution is used, these second moments do match. This can be seen by matching these second moments

$$\int_0^\infty \bar{\zeta}^{(0)}(t_0, \bar{r})\bar{r}^2\bar{r}\,d\bar{r} = \frac{C_0}{\tau_{20}}\int_0^\infty e^{-\eta^2}\bar{r}^2\bar{r}\,d\bar{r}\,.$$

From Eqs. (2.16) and (5.6), $C_0=\frac{1}{4}\pi$ and therefore

$$\tau_{20} = \frac{\pi}{2\Gamma^2}\int_0^\infty \bar{\zeta}^{(0)}(t_0, \bar{r})\bar{r}^2\bar{r}\,d\bar{r}\,,$$

which comparing with Eq. (5.4) shows that $\tau_{20}=\tau_{20}^*$. Therefore the one term optimum solution not only has the mathematical advantage of having $C_1=0$, but also has the physical advantage that its initial second moment matches the second moment of the given initial profile.

One term optimum solutions were computed for the example discussed in the next section. The results were within $\frac{1}{2}\%$ of the "exact" results shown in Figs. 3 and 4.

6. Example—A Pair of Coaxial Vortex Rings

In order to illustrate the results of the previous sections, a specific example is considered here. The following must be specified: (i) the initial vorticity distribution; (ii) the initial geometry of the ring; (iii) a given external inviscid flow in the absence of the ring.

The problem considered is that of two coaxial vortex rings, decaying and moving under mutual influence. The motion of a vortex pair was observed by Helmholtz (1867) and Reynolds (1876) and Lamb [1] gives a word description of their observations.

The initial geometry of the two rings is presented in Fig. 2. The initial vorticity distribution is assumed to be:

$$\bar{\zeta}_1^{(0)}(t_0, \bar{r}) = \begin{Bmatrix} 0 & \text{for} \quad \bar{r} > \bar{\delta}_{0i} \\ 2\Omega_i & \text{for} \quad \bar{r} < \bar{\delta}_{0i} \end{Bmatrix} \quad i=1, 2 \tag{6.1}$$

where $\bar{\delta}_{0i}=\delta_{0i}/\varepsilon$ and $\Omega_i=\Gamma_i/2\pi\bar{\delta}_{0i}^2$. Then from Eq. (3.10)

$$C_n = \frac{\tau_{10}^n}{4\pi}n!\sum_{j=0}^{n}\frac{(-1)^j 2^j}{(j+1)(j!)^2(n-j)!}\,. \tag{6.2}$$

In Eq. (6.2) and in what follows the subscript i is omitted. It is understood that in each case there are two such equations, one for each vortex. In deriving Eq. (6.2), τ_{10} was taken to be the optimum shift τ_{10}^*. From Eqs. (5.4) and (6.1):

$$\tau_{20}^* = \bar{\delta}_0^2/8\Gamma$$

and then Eq. (5.5) results in:

$$\tau_{10}^* = R_0(t_0)\bar{\delta}_0^2/8\Gamma\,.$$

The function $K(t)$ appearing in Eq. (2.13) can now be written explicitly using Eqs. (2.17), (3.8), (3.9), and (6.2):

$$K(t) = \ln[4\Gamma\tau_2]^{\frac{1}{2}} + 64\pi^2\int_0^\infty Q(\tau_1, \eta)S(\tau_1, \eta)\ln\eta\,d\eta \tag{6.3}$$

where

$$Q(\tau_1, \eta) = [\eta \bar{v}^{(0)}]_\eta = \bar{r}\bar{\zeta}^{(0)} = \eta\, e^{-\eta^2} \sum_{n=0}^{\infty} \frac{C_n}{\tau_1^n} L_n(\eta^2)$$

and

$$S(\tau_1, \eta) = \eta \bar{v}^{(0)} = \int_0^{\bar{r}} \bar{r}\bar{\zeta}^{(0)} d\eta = C_0 - e^{-\eta^2} \sum_{n=0}^{\infty} \frac{n!\, C_n}{\tau_1^n} \sum_{m=0}^{n} \frac{(-1)^m}{m!(n-m)!} \sum_{k=0}^{m} \frac{\eta^{2k}}{k!} .$$

The external inviscid flow for each vortex is wholly due to the other vortex. From Lamb [1], the inviscid stream function at a point (ρ, z) due to a vortex ring located at (R', Z') is given by

$$\Psi^* = -\frac{\Gamma}{2\pi}(r_1 + r_2)[K(\lambda) - E(\lambda)] \tag{6.4}$$

where $\lambda = (r_2 - r_1)/(r_1 + r_2)$, $r_{1,2}^2 = (z - Z')^2 + (\rho \mp R')^2$, and K and E are the complete elliptic integrals of the first and second kind, respectively. The velocity components are then given by

$$w_1^* = \frac{1}{r}\frac{\partial \Psi}{\partial z} \quad \text{and} \quad w_3^* = -\frac{1}{\rho}\frac{\partial \Psi}{\partial \rho},$$

or, using Eq. (6.4)

$$w_1^* = -\frac{\Gamma}{2\pi}\frac{(z - Z')}{\rho}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\left[K - \frac{1+\lambda^2}{1-\lambda^2} E\right] \tag{6.5a}$$

and:

$$w_3^* = \frac{\Gamma}{2\pi\rho}\left[\left(\frac{\rho - R'}{r_1} + \frac{\rho + R'}{r_2}\right)\left(K - \frac{E}{1-\lambda^2}\right) - \left(\frac{\lambda}{1-\lambda^2}\right)\left(\frac{\rho - R'}{r_1} - \frac{\rho + R'}{r_2}\right)E\right]. \tag{6.5b}$$

For the two vortex ring configuration, the label i identifies the particular ring as in Fig. 2 $(Z_{0_2}(t_0) > Z_{0_1}(t_0))$. Then for the vortex ring $i=1$, Eq. (6.5) is to be substituted into Eqs. (2.10) and (2.13) using the following parameters in Eq. (6.5):

$$\begin{matrix} \rho = R_{0_1}(t) & R' = R_{0_2}(t) \\ z = Z_{0_1}(t) & Z' = Z_{0_2}(t) \end{matrix} \quad \Gamma = \Gamma_2$$

For the vortex ring $i=2$, the correct parameters in Eq. (6.5) are:

$$\begin{matrix} \rho = R_{0_2}(t) & R' = R_{0_1}(t) \\ z = Z_{0_2}(t) & Z' = Z_{0_1}(t) . \end{matrix} \quad \Gamma = \Gamma_1$$

With the use of Eqs. (6.3) and (6.5), the right-hand sides of Eqs. (2.10) and (2.13) are known functions of τ_1, τ_2, R_0 and Z_0. Eqs. (2.10), (2.13) and (4.1) were integrated numerically for each member of the vortex pair for the standard initial geometry.

$$\left.\begin{matrix} R_{20}/R_{10} = Z_{20}/R_{10} = \Gamma_2/\Gamma_1 = 1 , \\ Z_{10} = 0 , \quad \delta_{0_1}/R_{0_1} = \delta_{0_2}/R_{0_2} = 1/100 \end{matrix}\right\} \text{at } t = t_0 , \quad \mathrm{Re} = 10^6$$

The resulting motions of the rings are presented in Fig. 3. The backward ring decreases in size $(\dot{R}_0 < 0)$ and overtakes the forward ring $(\dot{Z}_{0_1} > \dot{Z}_{0_2})$ which is increasing in size $(\dot{R}_{0_2} > 0)$. Then the ring that was initially the backward ring becomes the forward ring, and vice-versa, so that the roles are reversed. The rings therefore take turns overtaking each other and going through each other. This behavior is the one observed by Lamb [1].

Tung and Ting [2] derived an expression for the effective radius of the vortical core

$$\delta(t) = (4\nu\tau_2)^{\frac{1}{2}} . \tag{6.6}$$

The result from inviscid theory is

$$\delta_{\text{inv}} \sim (1/R)^{\frac{1}{2}} . \tag{6.7}$$

This last result is based on geometric arguments in conjunction with conservation of mass considerations. Fig. 4 displays graphs of Eqs. (6.6) and (6.7) for both vortex rings in the example being considered. In both cases, the core radius increases faster than predicted by inviscid theory. Moreover, at the end of one cycle inviscid theory predicts no change in the core radius, while the present theory predicts an increase in the core radii of both vortices. The latter result, which is due to viscous diffusion is the observed effect. (When the vortices return to a relative position similar to their initial relative position, they have completed one cycle. Figs. 3 and 4 are all plotted for one cycle). In Fig. 4 the net change in core radius for each vortex is indicated at the end of one cycle.

The results displayed in Figs. 3 and 4 were obtained by using enough terms in the series for $\bar{\zeta}^{(0)}$ (Eq. 3.9) to insure an accuracy of 0.01 % in $R_0(t)$, $Z_0(t)$, and $\tau_2(t)$. Results were also obtained using the optimum solution of section 5, and these results were within $\frac{1}{2}$% of the "exact" solutions shown in Figs. 3 and 4.

7. Concluding Remarks

A method of construction of solutions for the motion and decay of circular vortex rings has been presented. The initial vorticity distribution may be non-similar. Also presented is a method for introducing a time shift which makes the first term in the series solution for the vorticity be the "best" approximation. Finally, the analysis is applied to the motion and decay of a pair of coaxial vortex rings.

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